

DYNAMICAL TWISTS IN GROUP ALGEBRAS

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ABSTRACT. We classify dynamical twists in group algebras of finite groups. Namely, we set up a bijective correspondence between gauge equivalence classes of dynamical twists (which are solutions of a certain non-linear functional equation) and isomorphism classes of “dynamical data” described in purely group theoretical terms. This generalizes the classification of usual twists obtained by Movshev and Etingof-Gelaki.

1. INTRODUCTION

In the last few years, following the pioneering paper [F], the theory of quantum groups has developed a new branch – the theory of dynamical R -matrices and dynamical quantum groups (see [ES] for a review). In this theory, there is a useful notion of a dynamical twist, which was perhaps first introduced in [BBB].

Namely, let H be a Hopf algebra over a field k , and A be a finite Abelian group of group-like elements of H . A dynamical twist is an A -invariant function $A^* \rightarrow H \otimes H$, which satisfies a certain nonlinear functional equation (the so-called dynamical non-abelian 2-cocycle condition), see Definition 2.3.

If A is trivial, a dynamical twist is just an ordinary twist, in the sense of Drinfeld (see, e.g., [M]). Thus, the notion of a dynamical twist generalizes that of a usual twist.

The significance of the notion of a dynamical twist consists in the fact that given a dynamical twist $J(\lambda)$, one can endow the algebra $H \otimes \text{End}_k k[A]$ with a nontrivial structure of a weak Hopf algebra (see [EN]), which is quasitriangular if so is H . This is done using “twisting” of H by J , which is a procedure analogous to usual twisting of Hopf algebras. In particular, for a quasitriangular H with the R -matrix \mathcal{R} , the function $\mathcal{R}(\lambda) = J^{-1}(\lambda)^{21} \mathcal{R} J(\lambda)$ is a dynamical R -matrix, i.e., it satisfies the quantum dynamical Yang-Baxter equation (see [ES]).

In this paper we classify dynamical twists in group algebras of finite groups over an algebraically closed field k of characteristic zero. Since group algebras are trivially quasitriangular, all such twists give rise to dynamical R -matrices. Our classification generalizes the classification of usual twists in group algebras, which was done in [EG], following closely the paper [Mo].

Namely, let A be an abelian subgroup of a finite group G . A dynamical datum for (G, A) is a subgroup K of G together with a family of irreducible projective representations of K satisfying a certain coherence condition, see Definition 4.5. Our main result is the following

Theorem 6.6. There is a bijection between

- (i) gauge equivalence classes of dynamical twists, $J : A^* \rightarrow k[G] \otimes k[G]$,
- (ii) isomorphism classes of dynamical data for (G, A) .

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The structure of the paper is as follows.

In Section 2 we recall the definitions of a dynamical twist and dynamical gauge equivalence.

In Section 3 we use the idea of Movshev [Mo] to associate a family of semisimple algebras and bimodules with every dynamical twist. Here we also define the notions of minimal and minimizable dynamical twists.

The concept of a dynamical datum for a pair (G, A) is introduced in Section 4.

The main result of Section 5 is Theorem 5.3 which shows that every dynamical twist gives rise to an isomorphism class of dynamical data in such a manner that gauge equivalent twists give the same class of data.

The converse to this theorem is provided in Section 6, where we employ the exchange construction of [EV] to construct a gauge equivalence class of dynamical twists from a dynamical datum. The correspondence between classes of twists and classes of data is shown to be bijective in Theorem 6.6 which is the central result of this paper. Next, we explicitly construct an example of a dynamical datum leading to a non-minimizable dynamical twist (Example 6.9) – which is a purely “dynamical” phenomenon, that does not exist for usual twists. We also explicitly compute a family of dynamical twists in the group algebra of the group $F_p^\times \ltimes F_p$ of affine transformations of the line over the field F_p of p elements (Example 6.10).

Finally, in Section 7 we present a construction of minimal dynamical twists in group algebras of finite nilpotent groups. Our technique here uses the exponential map for nilpotent Lie algebras over finite fields and a result of Kazhdan [K].

We expect that the methods of this paper can be applied to constructing and classifying dynamical twists in universal enveloping algebras and, in particular, to reproving and possibly improving the main result of [X], stating that a splittable triangular dynamical r -matrix can be quantized. This is a subject of our future research.

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2. DYNAMICAL TWISTS IN HOPF ALGEBRAS

In this section we recall the definition of a dynamical twist [BBB].

Let H be a Hopf algebra and A be a finite Abelian subgroup of the group of group-like elements of H . Then $k[A] = \text{Map}(A^*, k)$ is a commutative and cocommutative Hopf subalgebra of H . Let P_μ , $\mu \in A^*$, be the minimal idempotents in $k[A]$ corresponding to characters of A .

Definition 2.1. For any left H -module X and character $\mu \in A^*$ define a *weight subspace*

$$(1) \quad X[\mu] = \{x \in X \mid ax = \mu(a)x, \text{ for all } a \in A\}.$$

Definition 2.2. We say that an element $x \in H^{\otimes n}$, $n \geq 1$ has *zero weight* if $x \in H^{\otimes n}[0]$, where $H^{\otimes n}$ is viewed as a left H -module via the adjoint action; in other words, if x commutes with $\Delta^n(a)$ for all $a \in A$, where $\Delta^n : k[A] \rightarrow k[A]^{\otimes n}$ is the iterated comultiplication.

Definition 2.3. Let $J(\lambda) : A^* \rightarrow H \otimes H$ be a zero weight function with invertible values. We say that $J(\lambda)$ is a *dynamical twist* in H if it satisfies the following functional equations

$$(2) \quad J^{12,3}(\lambda) J^{12}(\lambda - h^{(3)}) = J^{1,23}(\lambda) J^{23}(\lambda),$$

$$(3) \quad (\varepsilon \otimes \text{id})J(\lambda) = (\text{id} \otimes \varepsilon)J(\lambda) = 1.$$

Here $J^{12,3}(\lambda) = (\Delta \otimes \text{id})J(\lambda)$, $J^{12}(\lambda) = J(\lambda) \otimes 1$ etc. The notation $\lambda \pm h^{(i)}$ means that the argument λ is shifted by the weight of the i -th component, e.g., $J(\lambda - h^{(3)})$ is the element of $H \otimes H \otimes k[A]$ such that $\mu_3(J(\lambda - h^{(3)})) = J(\lambda - \mu)$, where the index i in μ_i indicates that μ is applied to the i th component.

Remark 2.4. There are two other versions of the dynamical twist condition (2) that appeared in the literature :

$$(4) \quad J^{12,3}(\lambda) J^{12}(\lambda + h^{(3)}) = J^{1,23}(\lambda) J^{23}(\lambda),$$

$$(5) \quad J^{12,3}(\lambda) J^{12}(\lambda + h^{(3)}) = J^{1,23}(\lambda) J^{23}(\lambda - h^{(1)}).$$

Note that (2) is obtained from equations (4) and (5) by the changes of variable $\lambda \mapsto -\lambda$ and $\lambda \mapsto \lambda + h^{(1)} + h^{(2)}$ respectively.

Definition 2.5. If $J(\lambda)$ is a dynamical twist in H and $t(\lambda) : A^* \rightarrow H$ is a zero weight function with invertible values such that $\varepsilon(t(\lambda)) \equiv 1$, then

$$(6) \quad J^t(\lambda) = \Delta(t(\lambda)^{-1}) J(\lambda) (t(\lambda - h^{(2)}) \otimes t(\lambda))$$

is also a dynamical twist in H , *gauge equivalent* to $J(\lambda)$. The function $t(\lambda)$ is called a *gauge transformation*.

Remark 2.6. Note ([ES], Appendix C) that a dynamical twist $J(\lambda)$ on H is completely defined by its value $J(\lambda_0)$, at any point $\lambda_0 \in A^*$. Indeed, from Equation (2) we have

$$(7) \quad J(\lambda_0 - \mu) = \mu_3(J^{12,3}(\lambda_0)^{-1} J^{1,23}(\lambda_0) J^{23}(\lambda_0)),$$

for all $\mu \in A^*$.

Remark 2.7. Let $H = k[G]$ be a group Hopf algebra and $J \in H \otimes H$ be an invertible element with the properties $(\varepsilon \otimes \text{id})J = (\text{id} \otimes \varepsilon)J = 1$ and

$$(J^{12,3})^{-1} J^{1,23} J^{23} \in k[G] \otimes k[G] \otimes k[A].$$

Then one can check by a direct computation ([ES], 12.1) that

$$(8) \quad J(\lambda) = -\lambda_3((J^{12,3})^{-1} J^{1,23} J^{23}).$$

is a dynamical twist in H .

Remark 2.8. If $A = \{1\}$ then the definition of dynamical twist coincides with the usual notion of twist introduced by Drinfeld.

Example 2.9. If $H = k[A]$, then H is commutative and can be identified with the algebra of functions on A^* . Let $P_\mu, \mu \in A^*$ be the minimal idempotents of H . Let $c : A^* \times A^* \rightarrow k^\times$ be any function such that $c(\lambda, 0) = c(0, \lambda) = 1$. Then $J = \sum_{\mu, \nu} c(\mu, \nu) P_\mu \otimes P_\nu$ satisfies the conditions of Remark 2.7 and hence

$$(9) \quad J(\lambda) = \sum_{\mu, \nu} c(\mu + \nu, -\lambda)^{-1} c(\mu, \nu - \lambda) c(\nu, -\lambda) P_\mu \otimes P_\nu.$$

is a dynamical twist. Furthermore, every dynamical twist in $k[A]$ is of this form by Remarks 2.6 and 2.7.

In fact, it turns out that in this case $J(\lambda)$ is always gauge equivalent to the constant twist $1 \otimes 1$. Namely, consider a gauge transformation

$$(10) \quad t(\lambda) = \sum_{\mu} c(\mu, -\lambda) P_\mu, \quad \lambda \in A^*,$$

then $J(\lambda) = \Delta(t(\lambda)^{-1})(t(\lambda) - h^{(2)}) \otimes t(\lambda)$.

3. ALGEBRAS B_λ ASSOCIATED WITH A DYNAMICAL TWIST

From now on let $H = k[G]$, the Hopf algebra of a finite group G , then A is a subgroup of G . Define $k[G/A]$ to be the quotient of $k[G]$ by the left ideal generated by the elements $(a - 1), a \in A$.

For any $\lambda \in A^*$ let us define a comultiplication and counit on $k[G/A]$ as follows

$$\Delta_\lambda(g) = (g \otimes g)J(\lambda), \quad \varepsilon_\lambda(g) = 1,$$

for all cosets $g \in G/A$. Note that since $J(\lambda)$ has zero weight, the above operations are well-defined.

Proposition 3.1. $C_\lambda = (k[G/A], \Delta_\lambda, \varepsilon_\lambda)$ is a coassociative G -coalgebra (where G acts via the left multiplication) with counit.

Proof. It follows from the dynamical twist identity (2) that

$$(11) \quad (\Delta \otimes \text{id})J(\lambda)(J(\lambda) \otimes 1) = (\text{id} \otimes \Delta)J(\lambda)(1 \otimes J(\lambda)) \text{ in } k[G] \otimes k[G] \otimes k[G/A]$$

which implies coassociativity of Δ_λ . The counit axiom is obvious. \square

Let B_λ be the associative algebra dual to C_λ . Then B_λ can be naturally identified with the algebra $F_0[G]$ of all functions on G invariant under right translations by elements of A with multiplication given by

$$(12) \quad f_1 *_\lambda f_2(g) = f_1 \otimes f_2((g \otimes g)J(\lambda)),$$

for all $f_1, f_2 \in F_0(G)$ and $g \in G$. It is a left G -algebra with the action of G given by

$$(13) \quad h \circ f(g) = f(h^{-1}g),$$

for all $f \in F_0[G]$ and $h, g \in G$.

Remark 3.2. If $t(\lambda)$ is a gauge transformation then coalgebras C_λ and C_λ^t (resp. algebras B_λ and B_λ^t) corresponding to $J(\lambda)$ and $J^t(\lambda)$ are isomorphic via

$$(14) \quad c \mapsto ct(\lambda), \quad c \in C_\lambda, \quad (\text{ resp. } f(g) \mapsto f(gt(\lambda)), \quad f \in B_\lambda, g \in G).$$

The statement and proof of the next Proposition are analogous to ([Mo], 7).

Proposition 3.3. B_λ is a semisimple algebra (equivalently, C_λ is a cosemisimple coalgebra).

Proof. Suppose B_λ is not semisimple and let I be the maximal non-zero power of its radical. Then I is G -stable and for all $z_1, z_2 \in I$ and $h_1, h_2 \in G$ we have $(g_1 \circ z_1)(g_2 \circ z_2) = 0$. The last condition is equivalent to

$$\langle \Delta_\lambda(c), (g_1 \circ z_1) \otimes (g_2 \circ z_2) \rangle = 0$$

for all $c \in C_\lambda$. In particular, we have

$$\langle (g_1 \otimes g_2)J(\lambda), z_1 \otimes z_2 \rangle = \langle (g_1 \otimes g_2)\Delta_\lambda(1), z_1 \otimes z_2 \rangle = 0,$$

but since $J(\lambda)$ is invertible in $k[G]^{\otimes 2}$, the elements $(g_1 \otimes g_2)J(\lambda) \pmod{A}$ span $k[G/A]$, so that $z_1 = z_2 = 0$, which is a contradiction. \square

Proposition 3.4. *G acts transitively on the set \mathcal{I}_λ of minimal two-sided ideals of B_λ . In particular, all minimal two-sided ideals of B_λ have the same dimension.*

Proof. Since $B_\lambda = F_0[G]$ as a G -module, the space of G -invariant elements of B_λ has dimension 1. For any orbit of G in \mathcal{I}_λ the corresponding central idempotent of B_λ is G -invariant, so there is a single orbit. \square

For every $\mu \in A^*$ consider the space $F_\mu[G]$ of all functions on G of weight μ ,

$$F_\mu[G] := \{f \in F[G] \mid f(ha) = f(h)\mu(a), \quad a \in A, h \in G\}.$$

Proposition 3.5. *$F_\mu[G]$ is a G -equivariant $B_{\lambda-\mu} - B_\lambda$ bimodule via*

$$\begin{aligned} f \circ f_\mu(g) &= f \otimes f_\mu((g \otimes g)J(\lambda)), & f \in B_{\lambda-\mu}, f_\mu \in F_\mu[G], \\ f_\mu \circ f'(g) &= f_\mu \otimes f'((g \otimes g)J(\lambda)), & f' \in B_\lambda, f_\mu \in F_\mu[G]. \end{aligned}$$

The actions of $B_{\lambda-\mu}$ and B_λ^{op} are faithful.

Proof. First, we check that $F_\mu[G]$ is a left $B_{\lambda-\mu}$ -module and right B_λ -module.

For all $f_1, f_2 \in B_{\lambda-\mu}$ we have, using the definition of $F_\mu[G]$ and dynamical twist equation (2) :

$$\begin{aligned} f_1 \circ (f_2 \circ f_\mu)(g) &= f_1 \otimes f_2 \otimes f_\mu((g \otimes g \otimes g)J^{1,23}(\lambda)J^{23}(\lambda)) \\ &= f_1 \otimes f_2 \otimes f_\mu((g \otimes g \otimes g)J^{12,3}(\lambda)J^{12}(\lambda - \mu)) \\ &= (f_1 *_{\lambda-\mu} f_2) \circ f_\mu(g). \end{aligned}$$

Also, for all $f'_1, f'_2 \in B_\lambda$ we have :

$$\begin{aligned} (f_\mu \circ f'_1) \circ f'_2(g) &= f_\mu \otimes f'_1 \otimes f'_2((g \otimes g \otimes g)J^{12,3}(\lambda)J^{12}(\lambda)) \\ &= f_\mu \otimes f'_1 \otimes f'_2((g \otimes g \otimes g)J^{1,23}(\lambda)J^{23}(\lambda)) \\ &= f_\mu \circ (f'_1 *_{\lambda} f'_2)(g). \end{aligned}$$

It is immediate from Equation (11) that the actions of $B_{\lambda-\mu}$ and B_λ commute.

To prove that the action of $B_{\lambda-\mu}$ is faithful suppose that $f \circ f_\mu = 0$ for some $f \in B_{\lambda-\mu}$ and all $f_\mu \in F_\mu[G]$. Then $J(\lambda)$ being invertible implies that $f \otimes f_\mu$ is identically equal to 0 on $k[G]^{\otimes 2}$, so that $f = 0$. The proof for the other action is completely similar. \square

Proposition 3.6. *All algebras B_λ are isomorphic to each other.*

Proof. By Proposition 3.4 all simple B_λ -modules have the same dimension, which we will denote d_λ .

Suppose $d_\lambda > d_\mu$ for some $\lambda, \mu \in A^*$. Let $N_\mu = \dim_k F_0[G]/d_\mu^2$ be the number of non-isomorphic simple B_μ -modules and $n_i, i = 1, \dots, N_\mu$ be the multiplicities with which they occur in the decomposition of $F_{\mu-\lambda}[G]$. By Proposition 3.5 B_λ^{op} is isomorphic to a subalgebra of the centralizer C_μ of B_μ in $\text{End}_k F_{\mu-\lambda}[G]$. But $C_\mu = \oplus_i M_{n_i}(k)$, therefore $n_i \geq d_\lambda$ for all $i = 1, \dots, N_\mu$. Then we have

$$\dim_k F_0[G] = \sum_i n_i d_\mu \geq N_\mu d_\lambda d_\mu > N_\mu d_\mu^2 = \dim_k F_0[G],$$

which is a contradiction. \square

Corollary 3.7. *$F_\mu[G]$ is isomorphic to the left regular $B_{\lambda-\mu}$ -module and to the right regular B_λ -module. In particular, $B_{\lambda-\mu}$ is naturally identified with the centralizer of B_λ^{op} in $\text{End}_k(F_\mu[G])$.*

Proof. In the proof of Proposition 3.6 we must have $n_i = d_\mu$, i.e., every simple B_λ -module has the multiplicity equal to its dimension. \square

Proposition 3.8. *For all $\lambda, \mu, \nu \in A^*$ the map*

$$\beta_{\mu\nu}^\lambda : F_\mu[G] \otimes_{B_\lambda} F_\nu[G] \rightarrow F_{\mu+\nu}[G]$$

defined by $\beta_{\mu\nu}^\lambda(f_\mu \otimes f_\nu)(g) = f_\mu \otimes f_\nu((g \otimes g)J(\lambda + \nu))$ for all $f_\mu \in F_\mu[G], f_\nu \in F_\nu[G]$, and $g \in G$ is an isomorphism of G -equivariant $B_{\lambda-\mu} - B_{\lambda+\nu}$ bimodules.

Proof. To show that each $\beta_{\mu\nu}^\lambda$ is well-defined, we compute

$$\begin{aligned} \beta_{\mu\nu}^\lambda(f_\mu \circ f \otimes f_\nu)(g) &= f_\mu \circ f \otimes f_\nu((g \otimes g)J(\lambda + \nu)) \\ &= f_\mu \otimes f \otimes f_\nu((g \otimes g \otimes g)J^{12,3}(\lambda + \nu)J^{12}(\lambda)) \\ &= f_\mu \otimes f \otimes f_\nu((g \otimes g \otimes g)J^{1,23}(\lambda + \nu)J^{23}(\lambda + \nu)) \\ &= f_\mu \otimes f \circ f_\nu((g \otimes g)J(\lambda + \nu)) \\ &= \beta_{\mu\nu}^\lambda(f_\mu \otimes f \circ f_\nu)(g) \end{aligned}$$

for all $f \in B_\lambda$. The third equality above uses Equation (2). It is clear that $\beta_{\mu\nu}^\lambda$ is a G -module homomorphism. To see that it is invertible we first observe that from Corollary 3.7 we have $\dim_k(F_\mu[G] \otimes_{B_\lambda} F_\nu[G]) = |G|/|A|$, so that the map in question is between spaces of the same dimension. So it suffices to show that it is surjective. But the range of $\beta_{\mu\nu}^\lambda$ clearly coincides with the range of the map given by the composition of the right translation by $J(\lambda)$ (which is surjective since $J(\lambda)$ is invertible) and the usual multiplication map $F_\mu[G] \otimes F_\nu[G] \rightarrow F_{\mu+\nu}[G]$ (which is surjective because there are invertible functions in each $F_\mu[G]$). \square

Definition 3.9. We say that a dynamical twist $J(\lambda)$ is *minimal* if B_λ is a simple algebra for some (and hence for all) $\lambda \in A^*$.

Remark 3.10. (i) The property of $J(\lambda)$ being minimal is invariant under gauge transformations.

(ii) If K is a subgroup of G containing A and $J(\lambda) : A^* \rightarrow k[K]^2$ is a dynamical twist, then it is obviously a dynamical twist in $k[G]$ which is minimal if and only if $K = G$ and $J(\lambda)$ is minimal in $k[K]$.

Definition 3.11. A dynamical twist $J(\lambda)$ in $k[G]$ is said to be *minimizable* if there exists a subgroup K of G containing A and a minimal dynamical twist $J'(\lambda) : A^* \rightarrow k[K] \otimes k[K]$ such that $J(\lambda)$ is gauge equivalent to $J'(\lambda)$, where the latter is regarded as a dynamical twist in $k[G]$.

As it was shown in [EG], all twists are minimizable if $A = \{1\}$. On the contrary, it is not always the case for $A \neq \{1\}$, moreover, it is not always true that A is conjugate to a subgroup of K , see Example 6.9. Also, $J(\lambda)$ is not necessarily minimizable even if $A = K$, see Example 6.10.

4. PROJECTIVE REPRESENTATIONS AND DYNAMICAL DATA

Definition 4.1. Let $c : G \times G \rightarrow k^\times$ be a 2-cocycle on G and $V \neq 0$ be a vector space over k . A *projective* representation of G on V with cocycle (Schur multiplier) c is a map $\pi : G \rightarrow GL(V)$ such that $\pi(g)\pi(h) = c(g, h)\pi(gh)$. A projective representation π is *linearizable* if c is a coboundary and *linear* if $c \equiv 1$. It is *irreducible* if $\pi(G)$ generates $\text{End}_k V$ as a vector space.

Definition 4.2. A *projective isomorphism* of two projective representations $\pi : G \rightarrow GL(V)$ and $\pi' : G \rightarrow GL(V')$ is an invertible k -linear map $\phi : V \rightarrow V'$ such that

$$(15) \quad \pi'(g) \circ \phi = \alpha(g)\phi \circ \pi(g), \quad g \in G$$

for some function $\alpha : G \rightarrow k$.

It is clear that α takes values in k^\times and that it is uniquely defined by ϕ . The cocycles of π and π' differ by the coboundary $\alpha(g)\alpha(h)\alpha(gh)^{-1}$.

Definition 4.3. We will call α the *multiplier* function for ϕ .

- Remark 4.4.* (a) Every projective representation of G descends to a homomorphism $G \rightarrow PGL(V)$ to the projective linear group of V .
 (b) Let V_i , $i = 1, 2$, be projective representations of G . Then $V_1 \otimes V_2$ is a projective representation of G in the obvious way. If the cocycles of V_1 and V_2 are the same, then $V_1 \oplus V_2$ is also a projective representation of G .
 (c) For every 2-cocycle c on G there exists a canonical central extension \widehat{G} of G by a subgroup L of k^\times in which the cocycle c takes values :

$$(16) \quad 1 \longrightarrow L \longrightarrow \widehat{G} \xrightarrow{p} G \longrightarrow 1.$$

Explicitly, the multiplication in \widehat{G} is given by $(g, x)(h, y) = (gh, c(g, h)xy)$, for all $g, h \in G$ and $x, y \in L$. Any projective representation $\pi : G \rightarrow GL(V)$ with Schur multiplier c canonically lifts to a linear representation $\widehat{\pi} : \widehat{G} \rightarrow GL(V)$, via $\widehat{\pi}(g, x) = x\pi(g)$. In this linear representation $\widehat{\pi}$ the elements of $L \subset k^\times$ act on V by scalar multiplication.

- (d) Projectively isomorphic representations of G with the same Schur multiplier are not necessarily isomorphic as linear representations of \widehat{G} even for $c = 1$. For example, any one-dimensional representation $\chi : G \rightarrow k^\times$ is projectively isomorphic to the trivial representation of G .

We are ready to define the main object of this paper.

Definition 4.5. A *dynamical datum* for a pair (G, A) consists of a subgroup K of G and a family of projective irreducible representations $\pi_\lambda : K \rightarrow GL(V_\lambda)$, $\lambda \in A^*$, such that $V_{\lambda-\mu} \otimes V_\lambda^*$ is linear and

$$(17) \quad \text{Ind}_K^G(V_\lambda \otimes V_{\lambda-\mu}^*) \cong \text{Ind}_A^G \mu$$

for all $\lambda, \mu \in A^*$.

Remark 4.6. Note that Definition 4.5 in particular implies that the 2-cocycles (Schur multipliers) of all V_λ 's are equal, so that there is a canonical common central extension

$$(18) \quad 1 \longrightarrow L \longrightarrow \widehat{K} \longrightarrow K \longrightarrow 1,$$

linearizing all π_λ , $\lambda \in A^*$.

Proposition 4.7. *For any dynamical datum, V_λ and V_μ are non-equivalent representations of \widehat{K} for $\lambda \neq \mu$.*

Proof. From Definition 4.5 we have $\text{Hom}_{\widehat{K}}(V_\lambda, V_\mu \otimes X) = X[\lambda - \mu]$ for any G -module X (regarded as a \widehat{K} -module on which the kernel L acts trivially), where $X[\lambda] \subset X$ is a weight subspace of X defined in 2.1. Taking the trivial G -module $X = k$ we get $\text{Hom}_{\widehat{K}}(V_\lambda, V_\mu) = 0$. \square

Remark 4.8. Nevertheless, V_λ and V_μ , $\lambda \neq \mu$, can be projectively isomorphic as projective representations of K , e.g., when they are 1-dimensional.

Definition 4.9. Let K and K' be two subgroups of G . An *isomorphism* of dynamical data $(K, \{V_\lambda \mid \lambda \in A^*\})$ and $(K', \{V'_\lambda \mid \lambda \in A^*\})$ is an element $g \in G$ such that $\text{Ad}_g K = K'$, where Ad_g is the adjoint action of g on G , and a family of projective isomorphisms $\phi_\lambda : V_\lambda \cong V'_\lambda$, $\lambda \in A^*$ of projective representations π_λ and $\pi'_\lambda \circ \text{Ad}_g$, having the same multiplier function $\alpha(h) = \alpha_\lambda(h)$, $h \in K$.

Definition 4.10. A dynamical datum is called *minimal* if $K = G$.

5. A DYNAMICAL DATUM DEFINED BY DYNAMICAL TWIST

In this section for every dynamical twist $J(\lambda)$ we define an isomorphism class of dynamical data, cf. Definitions 4.5 and 4.9. We use the notation of Section 3.

Proposition 5.1. *Let \mathcal{I}_λ be the set of minimal 2-sided ideals of B_λ . There are isomorphisms of G -sets $\tau_{\lambda\mu} : \mathcal{I}_\lambda \rightarrow \mathcal{I}_\mu$ such that $\tau_{\lambda\mu} \circ \tau_{\mu\nu} = \tau_{\lambda\nu}$.*

In other words, we can simultaneously identify all \mathcal{I}_λ 's as G -sets, $\mathcal{I}_\lambda \cong \mathcal{I}$.

Proof. Observe that a decomposition of $F_{\mu-\lambda}[G]$ into the sum of simple $B_\mu - B_\lambda$ -bimodules establishes an isomorphism of G -sets $\tau_{\lambda\mu} : \mathcal{I}_\lambda \rightarrow \mathcal{I}_\mu$. But the relative tensor product of bimodules $F_{\mu-\lambda}[G]$ and $F_{\lambda-\nu}[G]$ is isomorphic to $F_{\mu-\nu}[G]$ by Proposition 3.8, whence the composition rule of τ 's follows. \square

Thus, we can canonically identify stabilizers of points of \mathcal{I}_λ . Let us fix a point $i \in \mathcal{I}$ and let $K \subset G$ be the stabilizer subgroup of i and $B_\lambda^i \subset B_\lambda$ be the minimal ideal in \mathcal{I}_λ corresponding to i . Let V_λ be a (unique up to an isomorphism) simple

B_λ^i -module, then K acts on $\text{End}_k V_\lambda$ by automorphisms and we can choose an irreducible projective representation

$$(19) \quad \pi_\lambda : K \rightarrow GL(V_\lambda)$$

such that the action of $g \in K$ is given by $\text{Ad}_{\pi_\lambda(g)}$.

Proposition 5.2 (cf. [Mo], 8). *Projective representations π_λ are irreducible.*

Proof. We need to show that the space of K -invariant elements of B_λ^i has dimension 1. Any such an element injectively corresponds to a G -invariant element of B_λ , but the latter is equivalent to $k[G/A]$ as a left G -module, for which the space of invariants is 1-dimensional. \square

Observe that the K -equivariant $B_\mu^i - B_\nu^i$ bimodule $F_{\mu\nu}^i = B_\mu^i F_{\nu-\mu}[G] B_\nu^i$ is such that $F_{\nu-\mu}[G] \cong k[G] \otimes_{k[K]} F_{\mu\nu}^i$, i.e., $\text{Ind}_K^G F_{\mu\nu}^i \cong \text{Ind}_A^G(\mu - \nu)$.

Theorem 5.3. *Every dynamical twist $J(\lambda) : A^* \rightarrow k[G] \otimes k[G]$ defines an isomorphism class of dynamical data. Moreover, any two gauge equivalent dynamical twists $J(\lambda)$ and $\tilde{J}(\lambda)$ define the same isomorphism class of data.*

Proof. Given a dynamical twist $J(\lambda)$ fix a point $i \in \mathcal{I}$ and a subgroup $K \subset G$ as above. Consider the corresponding algebras B_λ^i and $B_\lambda^i - B_\mu^i$ bimodules $F_{\lambda\mu}^i$ and fix simple B_λ^i -modules $V_\lambda := V_\lambda^i$. Let $\rho_{\lambda\lambda} : F_{\lambda\lambda} \rightarrow \text{End}_k V_\lambda$ be the isomorphisms identifying K -algebras and choose bimodule isomorphisms $\rho_{\lambda\mu} : F_{\lambda\mu}^i \rightarrow V_\lambda \otimes V_\mu^*$. Let $\text{Sh}_{\lambda\mu}(g)$ denote the action of $g \in K$ on $F_{\lambda\mu}^i \subset F_{\mu-\lambda}[G]$ by shifts, cf. Equation (13). For every λ choose an irreducible projective representation $\pi_\lambda : K \rightarrow GL(V_\lambda)$ such that

$$(20) \quad \pi_\lambda(g) \otimes \pi_\lambda^*(g)^{-1} = \rho_{\lambda\lambda}^{-1} \circ \text{Sh}_{\lambda\lambda}(g) \circ \rho_{\lambda\lambda}.$$

Then for all $x \in F_{\lambda\mu}^i$, $f \in B_\lambda^i$, $f' \in B_\mu^i$ we have

$$\begin{aligned} & (\tau_{\lambda\mu} \circ \text{Sh}_{\lambda\mu}(g)^{-1} \circ \rho_{\lambda\mu}^{-1})(\pi_\lambda(g) \otimes \pi_\mu^*(g)^{-1})(f \circ x \circ f') = \\ & = f \circ (\rho_{\lambda\mu} \circ \text{Sh}_{\lambda\mu}(g)^{-1} \circ \rho_{\lambda\mu}^{-1}) \circ (\pi_\lambda(g) \otimes \pi_\mu^*(g)^{-1})x \circ f', \end{aligned}$$

i.e., $(\rho_{\lambda\mu} \circ \text{Sh}_{\lambda\mu}(g)^{-1} \circ \rho_{\lambda\mu}^{-1}) \circ (\pi_\lambda(g) \otimes \pi_\mu^*(g)^{-1})$ commutes with the action of $B_\lambda^i \otimes (B_\mu^i)^{op}$ on $F_{\lambda\mu}^i$. Therefore, by Schur's Lemma,

$$(21) \quad \pi_\lambda(g) \otimes \pi_\mu^*(g)^{-1} = \gamma_{\lambda\mu}(g) \rho_{\lambda\mu}^{-1} \circ \text{Sh}_{\lambda\mu}(g) \circ \rho_{\lambda\mu}$$

for some functions $\gamma_{\lambda\mu} : K \rightarrow k^\times$. By Proposition 3.8, $F_{\lambda\mu}^i \otimes_{B_\mu^i} F_{\mu\nu}^i \cong F_{\lambda\nu}^i$ as K -equivariant bimodules, whence $\gamma_{\lambda\mu}(g)\gamma_{\mu\nu}(g) = \gamma_{\lambda\nu}(g)$ for all λ, μ, ν . Replacing $\pi_\lambda(g)$ by a projectively isomorphic representation $\pi'_\lambda(g) = a_\lambda(g)\pi_\lambda(g)$, where $a_\lambda(g)^{-1}a_\mu(g) = \gamma_{\lambda\mu}(g)$, we obtain a collection of irreducible projective representations π'_λ of K such that $V_\lambda \otimes V_\mu^*$ is linear and induces $F_{\mu-\lambda}[G] \cong \text{Ind}_A^G(\lambda - \mu)$, i.e., a dynamical datum for (G, A) .

Let us show that gauge equivalent twists define isomorphic dynamical data, for any choice of projective representations π_λ . This will imply, in particular, that the isomorphism class of a dynamical datum we have constructed above is well defined.

Let $\tilde{J}(\lambda)$ be a dynamical twist in $k[G]$ gauge equivalent to $J(\lambda)$ via a gauge transformation $t(\lambda)$. By Remark 3.2 the map $t_\lambda : f(g) \mapsto f(gt(\lambda))$ defines an isomorphism between the corresponding G -algebras B_λ and \tilde{B}_λ . This establishes a bijective correspondence between minimal ideals of B_λ^i and \tilde{B}_λ^i of these algebras

and allows to identify their stabilizers, so we can assume that both $J(\lambda)$ and $\tilde{J}(\lambda)$ define the same subgroup K .

Let $\tilde{F}_{\lambda\mu}^i$ be the corresponding $\tilde{B}_\lambda^i - \tilde{B}_\mu^i$ bimodules. Fix vector spaces $\tilde{V}_\lambda := \tilde{V}_\lambda^i$ and identify K -algebras \tilde{B}_λ^i and $\text{End}_k \tilde{V}_\lambda$. Choose isomorphisms $\tilde{\rho}_{\lambda\mu} : \tilde{F}_{\lambda\mu}^i \rightarrow \tilde{V}_\lambda \otimes \tilde{V}_\mu^*$ of bimodules, and projective representations $\tilde{\pi}_\lambda : K \rightarrow GL(\tilde{V}_\lambda)$ such that

$$\tilde{\pi}_\lambda(g) \otimes \tilde{\pi}_\lambda^*(g)^{-1} = \tilde{\rho}_{\lambda\lambda}^{-1} \circ \tilde{\text{Sh}}_{\lambda\lambda}(g) \circ \tilde{\rho}_{\lambda\lambda},$$

where $\tilde{\text{Sh}}_{\lambda\lambda}(g)$ denotes the action of g on $\tilde{F}_{\lambda\lambda}^i$. Arguing as above, we get functions $\tilde{\gamma}_{\lambda\mu}(g)$ and $\tilde{a}_\lambda(g)$ such that

$$(22) \quad \tilde{\pi}_\lambda(g) \otimes \tilde{\pi}_\mu^*(g)^{-1} = \tilde{\gamma}_{\lambda\mu}(g) \tilde{\rho}_{\lambda\mu}^{-1} \circ \tilde{\text{Sh}}_{\lambda\mu}(g) \circ \tilde{\rho}_{\lambda\mu},$$

and $\tilde{\pi}'_\lambda(g) = \tilde{a}_\lambda(g) \tilde{\pi}_\lambda(g)$ (where $\tilde{a}_\lambda(g) \tilde{a}_\mu(g)^{-1} = \tilde{\gamma}_{\lambda\mu}(g)^{-1}$) make $\tilde{V}_\lambda \otimes \tilde{V}_\mu^*$ linear.

Isomorphisms between simple K -algebras B_λ^i and \tilde{B}_λ^i yield projective isomorphisms ϕ_λ (defined up to scaling) between π_λ and $\tilde{\pi}_\lambda$, $\lambda \in A^*$:

$$(23) \quad \tilde{\pi}_\lambda(g) \circ \phi_\lambda = \alpha_\lambda(g) \phi_\lambda \circ \pi_\lambda(g), \quad g \in K,$$

for some multipliers $\alpha_\lambda : K \rightarrow k^\times$.

Comparing Equations (21) and (22) and using (23) we conclude that

$$\frac{\tilde{\gamma}_{\lambda\mu}(g)}{\gamma_{\lambda\mu}(g)} = \frac{\alpha_\lambda(g)}{\alpha_\mu(g)}, \quad g \in K.$$

We have $\tilde{\pi}'_\lambda(g) \circ \phi_\lambda = \alpha'_\lambda(g) \phi_\lambda \circ \pi'_\lambda(g)$, where $\alpha'_\lambda(g) = \alpha_\lambda(g) \frac{\tilde{a}_\lambda(g)}{a_\lambda(g)}$ and

$$\frac{\alpha'_\lambda(g)}{\alpha'_\mu(g)} = \frac{\alpha_\lambda(g)}{\alpha_\mu(g)} \frac{\tilde{a}_\lambda(g)}{\tilde{a}_\mu(g)} \frac{a_\mu(g)}{a_\lambda(g)} = 1,$$

so that all multipliers α'_λ are equal, i.e., dynamical data constructed from $J(\lambda)$ and $\tilde{J}(\lambda)$ are isomorphic. \square

Corollary 5.4. *The above construction assigns minimal dynamical data to minimal dynamical twists.*

Proof. This is clear since B_λ is simple if and only if $G = K$. \square

Remark 5.5. It follows from the definition of a dynamical datum that for every $\mu \in A^*$ the weight subspace $F_\mu[G]$ can be regarded as the space of functions on G/K with values in $\text{Hom}_k(V_\lambda, V_{\lambda-\mu})$, or, equivalently, as the space of K -homomorphisms from $k[G]$ to $\text{Hom}_k(V_\lambda, V_{\lambda-\mu})$, where $h \in K$ acts on $k[G]$ by the right multiplication by h^{-1} and by conjugation on $\text{Hom}_k(V_\lambda, V_{\lambda-\mu})$.

With this identification the map $F_\nu[G] \otimes_{B_{\lambda-\mu}} F_\mu[G] \rightarrow F_{\nu+\mu}[G]$ defined in Proposition 3.8 is given by the composition

$$(24) \quad \begin{aligned} k[G] &\rightarrow k[G] \otimes k[G] \rightarrow \text{Hom}_k(V_{\lambda-\mu}, V_{\lambda-\mu-\nu}) \otimes \text{Hom}_k(V_\lambda, V_{\lambda-\mu}) \rightarrow \\ &\rightarrow \text{Hom}_k(V_\lambda, V_{\lambda-\mu-\nu}), \end{aligned}$$

where the first map is the comultiplication $g \rightarrow g \otimes g$, the second uses the above description of the weight subspaces, and the third is the composition of homomorphisms.

6. CONSTRUCTION OF A DYNAMICAL TWIST FROM DYNAMICAL DATUM

We use the exchange construction that appeared in [EV] (see also [ES]) to produce a dynamical twist from a dynamical datum for (G, A) .

Given a dynamical datum $(K, \{V_\lambda \mid \lambda \in A^*\})$ as in Definition 4.5 let \widehat{K} be the canonical central extension of K linearizing all V_λ . Then every G -module X is also a \widehat{K} -module on which the kernel of the extension (which is a central subgroup of \widehat{K}) acts trivially.

Choose isomorphisms $\epsilon_{\lambda\mu} : V_\lambda \otimes V_{\lambda-\mu}^* \cong \text{Ind}_A^G \mu$ such that for $\lambda = \mu$ the identity of $\text{End}_k V_\lambda$ is mapped to 1. Using the definition of dynamical datum and Frobenius reciprocity for induced modules we have

$$(25) \quad \begin{aligned} \text{Hom}_{\widehat{K}}(V_\lambda, V_{\lambda-\mu} \otimes X) &\cong \text{Hom}_K(V_\lambda \otimes V_{\lambda-\mu}^*, X) \\ &\cong \text{Hom}_G(\text{Ind}_A^G \mu, X) \cong X[\mu], \end{aligned}$$

where the second isomorphism is defined by $\epsilon_{\lambda\mu}$ and the other two isomorphisms are canonical.

Let us denote $\Psi(\lambda, x)$ the homomorphism $V_\lambda \rightarrow V_{\lambda-\mu} \otimes X$ determined by $x \in X[\mu]$ through the above sequence of isomorphisms. Let Y be another G -module and $y \in Y[\nu]$, $\nu \in A^*$. Then

$$(26) \quad (\Psi(\lambda - \mu, y) \otimes \text{id}) \circ \Psi(\lambda, x) : V_\lambda \rightarrow V_{\lambda-\mu-\nu} \otimes Y \otimes X$$

corresponds to a unique element in $(Y \otimes X)[\mu + \nu]$. We introduce a linear operator $J_{Y,X}(\lambda) : Y \otimes X \rightarrow Y \otimes X$ by setting

$$(27) \quad (\Psi(\lambda - \mu, y) \otimes \text{id}) \circ \Psi(\lambda, x) := \Psi(\lambda, J_{Y,X}(\lambda)(y \otimes x)).$$

We will show that the function $J(\lambda) \in k[G] \otimes k[G]$ defined by this equation is a dynamical twist for $k[G]$.

Remark 6.1. Note that the above defined $J(\lambda)$ is related to the composition map

$$\text{Hom}_k(V_{\lambda-\mu}, V_{\lambda-\mu-\nu}) \otimes \text{Hom}_k(V_\lambda, V_{\lambda-\mu}) \rightarrow \text{Hom}_k(V_\lambda, V_{\lambda-\mu-\nu}),$$

in the following way. Take $f_\mu \in F_\mu[G]$, $f_\nu \in F_\nu[G]$. If we regard $\Psi(\lambda, f_\mu)$ as a K -homomorphism from $k[G]$ to $\text{Hom}_k(V_\lambda, V_{\lambda-\mu})$ (where $h \in K$ acts on $k[G]$ as $h \cdot g = gh^{-1}$) then

$$(28) \quad \Psi(\lambda, J(\lambda)(f_\nu \otimes f_\mu)) = \text{comp}(\Psi(\lambda - \mu, f_\nu) \otimes \Psi(\lambda, f_\mu)) \circ \Delta,$$

where $\Delta : k[G] \rightarrow k[G] \otimes k[G]$ is the comultiplication and $\text{comp}()$ is the composition of homomorphisms (cf. Equation (24)).

Lemma 6.2. *$J(\lambda)$ has zero weight.*

Proof. For all G -modules X, Y and elements $x \in X[\mu]$, $y \in Y[\nu]$, and $a \in A$ we have :

$$J_{Y,X}(\lambda)(ay \otimes ax) = (\mu + \nu)(a)J_{Y,X}(\lambda)(y \otimes x) = (a \otimes a)J_{Y,X}(\lambda)(y \otimes x),$$

since $J_{Y,X}(\lambda)$ maps $Y[\nu] \otimes X[\mu]$ to $(Y \otimes X)[\mu + \nu]$. \square

Lemma 6.3 (cf. [Mo]). *$J(\lambda)$ is invertible for all $\lambda \in A^*$.*

Proof. We need to show that

$$J(\lambda) : \bigoplus_{\mu} Y[\mu - \eta] \otimes X[\lambda - \mu] \rightarrow (Y \otimes X)[\lambda - \eta]$$

is surjective for all $\lambda, \eta \in A^*$, where X, Y are G -modules. Equivalently, we need to prove that the composition map

$$\bigoplus_{\mu} \text{Hom}_{\widehat{K}}(V_{\lambda} \otimes X^*, V_{\mu}) \otimes \text{Hom}_{\widehat{K}}(V_{\mu}, V_{\eta} \otimes Y) \rightarrow \text{Hom}_{\widehat{K}}(V_{\lambda} \otimes X^*, V_{\eta} \otimes Y)$$

is surjective. But this follows from the fact that $V_{\eta} \otimes Y \cong \bigoplus_{\mu} Y[\mu - \eta] \otimes V_{\mu}$ as \widehat{K} -modules. Indeed, $\text{Hom}_{\widehat{K}}(V_{\mu}, V_{\eta} \otimes Y) \cong Y[\mu - \eta]$ by Equation (25) and V_{μ} 's are mutually non-equivalent by Proposition 4.7. Therefore, a copy of $\bigoplus_{\mu} Y[\mu - \eta] \otimes V_{\mu}$ is contained in $V_{\eta} \otimes Y$, and has the same dimension. \square

Theorem 6.4. *$J(\lambda)$ is a dynamical twist for $k[G]$. Moreover, two isomorphic dynamical data $(K, \{V_{\lambda} \mid \lambda \in A^*\})$ and $(K', \{V'_{\lambda} \mid \lambda \in A^*\})$ define the same gauge equivalence class of dynamical twists.*

Proof. We have already seen that $J(\lambda)$ is an invertible zero weight function, we need to show that it satisfies the twist properties of Definition 2.3. Let X, Y, Z be G -modules and $x \in X[\mu], y \in Y[\nu], z \in Z[\eta]$. Consider the composition of \widehat{K} -module homomorphisms

$$V_{\lambda} \rightarrow V_{\lambda-\mu} \otimes X \rightarrow V_{\lambda-\mu-\nu} \otimes Y \otimes X \rightarrow V_{\lambda-\mu-\nu-\eta} \otimes Z \otimes Y \otimes X$$

defined by x, y, z . The associativity law gives two different ways of factorizing it:

$$\begin{aligned} & (\Psi(\lambda - \mu, J_{Z,Y}(\lambda - \mu)(z \otimes y)) \otimes \text{id}) \circ \Psi(\lambda, x) = \\ & = (\Psi(\lambda - \mu, z) \otimes \text{id}) \circ \Psi(\lambda, J_{Y,X}(\lambda)(y \otimes x)), \end{aligned}$$

whence we have $J_{Z \otimes Y, X}(\lambda)(J_{Z,Y}(\lambda - \mu) \otimes \text{id}) = J_{Z,Y \otimes X}(\lambda)(\text{id} \otimes J_{Y,X}(\lambda))$, i.e.,

$$J^{12,3}(\lambda) J^{12}(\lambda - h^{(3)}) = J^{1,23}(\lambda) J^{23}(\lambda)$$

for all λ . The counital properties of $J(\lambda)$ are clear since $\Psi(\lambda, \varepsilon) = \text{id}_{V_{\lambda}}$.

In proving the gauge equivalence of dynamical twists coming from isomorphic data we may assume that $K = K'$. Let $\phi_{\lambda} : V_{\lambda} \rightarrow V'_{\lambda}$, $\lambda \in A^*$ be an isomorphism between two dynamical data, $\Psi(\lambda, x), \Psi'(\lambda, x)$ ($x \in X[\mu]$) be associated homomorphisms of modules, and $J(\lambda), J'(\lambda)$ the dynamical twists. Define $t_X(\lambda) : X \rightarrow X$ by the identity

$$\Psi'(\lambda, t_X(\lambda)x) = (\phi_{\lambda-\mu} \otimes \text{id}) \circ \Psi(\lambda, x) \circ \phi_{\lambda}^{-1}.$$

Note that $t_X(\lambda)$ is well defined since all ϕ_{λ} have the same multiplier function, cf. Definition 4.9. Then $t_X(\lambda)$ is a zero weight function taking invertible values and satisfying $\varepsilon(t_X(\lambda)) = 1$ since $t_k(\lambda) = \text{id}_k$. For any $y \in Y[\nu]$ we compute

$$\begin{aligned} & \Psi'(\lambda, J'_{Y,X}(\lambda)(y \otimes x)) = \\ & = (\Psi'(\lambda - \mu, y) \otimes \text{id}) \circ \Psi'(\lambda, x) \\ & = (\phi_{\lambda-\mu-\nu} \otimes \text{id}) \circ \Psi(\lambda - \mu, t_Y(\lambda - \mu)y) \circ \Psi(\lambda, t_X(\lambda)(x)) \circ \phi_{\lambda}^{-1} \\ & = (\phi_{\lambda-\mu-\nu} \otimes \text{id}) \circ \Psi(\lambda, J_{Y,X}(\lambda)(t_Y(\lambda - \mu)y \otimes t_X(\lambda)x)) \circ \phi_{\lambda}^{-1} \\ & = \Psi'(\lambda, t_{Y \otimes X}(\lambda)^{-1} J_{Y,X}(\lambda)(t_Y(\lambda - \mu)y \otimes t_X(\lambda)x)), \end{aligned}$$

therefore $J'_{Y,X}(\lambda) = t_{Y \otimes X}(\lambda)^{-1} J_{Y,X}(\lambda) (t_Y(\lambda - \mu) \otimes t_X(\lambda))$, and

$$J'(\lambda) = \Delta(t(\lambda))^{-1} J(\lambda) (t(\lambda - h^{(2)}) \otimes t(\lambda)),$$

i.e., $J'(\lambda)$ is a gauge transformation of $J(\lambda)$. \square

Remark 6.5. One cannot canonically construct a concrete $J(\lambda)$ from $(K, \{V_\lambda\})$ or vice versa. It is only possible to assign an equivalence class of dynamical twists to every $(K, \{V_\lambda\})$ and isomorphism class of data to every $J(\lambda)$.

Theorem 6.6. *The maps D and T between gauge equivalence classes of twists and isomorphism classes data described by constructions of Section 5 and the beginning of this Section, respectively,*

$$(29) \quad \left\{ \begin{array}{c} \text{gauge equivalence} \\ \text{classes of} \\ \text{dynamical twists} \\ J(\lambda) : A^* \rightarrow k[G] \otimes k[G] \end{array} \right\} \begin{array}{c} \xrightarrow{D} \\ \xleftarrow{T} \end{array} \left\{ \begin{array}{c} \text{isomorphism} \\ \text{classes of} \\ \text{dynamical data} \\ (K, \{V_\lambda \mid \lambda \in A^*\}) \end{array} \right\},$$

are inverses of each other, i.e., define a bijective correspondence between the two sets involved.

Proof. Let $J(\lambda)$ be a dynamical twist, $(K, \{V_\lambda\})$ be (a representative of the class of) dynamical data associated to $J(\lambda)$, and $\tilde{J}(\lambda)$ be a dynamical twist coming from the exchange construction for $(K, \{V_\lambda\})$. According to Remarks 5.5 and 6.1 (cf. Equations (24) and (28)) both $J(\lambda)$ and $\tilde{J}(\lambda)$ are determined by the map

$$F_\nu[G] \otimes_{B_{\lambda-\nu}} F_\mu[G] \rightarrow F_{\nu+\mu}[G]$$

corresponding to the composition

$$\text{Hom}_k(V_{\lambda-\mu}, V_{\lambda-\mu-\nu}) \otimes \text{Hom}_k(V_\lambda, V_{\lambda-\mu}) \rightarrow \text{Hom}_k(V_\lambda, V_{\lambda-\mu-\nu}).$$

Comparing two G -module isomorphisms $\text{Hom}_K(k[G], \text{Hom}_k(V_\lambda, V_{\lambda-\mu})) \cong F_\mu[G]$ (the one established in Theorem 5.3, corresponding to the construction of $\{V_\lambda\}$ from $J(\lambda)$, and the one used in the exchange construction above) we get a zero weight function with invertible values $t(\lambda) : A^* \rightarrow k[G]$ that implements the corresponding G -module automorphism

$$F_\mu[G] \rightarrow F_\mu[G] : f(g) \mapsto f(gt(\lambda)), \quad g \in G,$$

for all λ , and is such that

$$f_\nu \otimes f_\mu((gt(\lambda - \mu) \otimes gt(\lambda))\tilde{J}(\lambda)) = f_\nu \otimes f_\mu((g \otimes g)J(\lambda)\Delta(t(\lambda))),$$

for all $f_\nu \in F_\nu[G]$, $f_\mu \in F_\mu[G]$. Therefore $J(\lambda)$ and $\tilde{J}(\lambda)$ are gauge equivalent, i.e., $T \circ D = \text{id}$, in particular T is surjective.

Let us show that T is also injective. Let $(K, \pi_\lambda : K \rightarrow GL(V_\lambda))$ and $(\tilde{K}, \tilde{\pi}_\lambda : \tilde{K} \rightarrow GL(\tilde{V}_\lambda))$ be two sets of dynamical data that produce dynamical twists $J(\lambda)$ and $\tilde{J}(\lambda)$ gauge equivalent to each other. By Remark 3.2 their G -algebras B_λ and \tilde{B}_λ are isomorphic via

$$f(g) \mapsto f(gt(\lambda)), \quad f \in B_\lambda = F_0[G].$$

Also, for the corresponding bimodules $F_{\lambda\mu}$ and $\tilde{F}_{\lambda\mu}$ it follows from the exchange construction of a dynamical twist in the beginning of this section that there are canonical isomorphisms of G -modules

$$F_{\lambda\mu} \cong (\text{Hom}_k(V_\lambda, V_{\lambda-\mu}) \otimes F[G])^K, \quad (\text{resp. } \tilde{F}_{\lambda\mu} \cong (\text{Hom}_k(\tilde{V}_\lambda, \tilde{V}_{\lambda-\mu}) \otimes F[G])^{\tilde{K}}),$$

where K (resp. \tilde{K}) acts by right translations on $F[G]$ and in a standard way on the Hom space, whereas $g \in G$ acts on $F[G]$ by the left translation by g^{-1} and trivially on the first factor.

For $\lambda = \mu$ this implies that all the matrix blocks of B_λ (resp. \tilde{B}_λ) are canonically isomorphic to $\text{End}_k V_\lambda$ (resp. $\text{End}_k \tilde{V}_\lambda$) and form a G -homogeneous space isomorphic to G/K (resp. G/\tilde{K}). Therefore \tilde{K} is conjugate to K and we can assume $\tilde{K} = K$.

Thus, we have a canonical K -algebra isomorphism $t_\lambda : \text{End}_k V_\lambda \rightarrow \text{End}_k \tilde{V}_\lambda$ and so there is a projective isomorphism $\phi_\lambda : V_\lambda \rightarrow \tilde{V}_\lambda$ such that $\text{Ad} \phi_\lambda = t_\lambda$. Note that ϕ_λ is defined up to a scalar, therefore its multiplier function $\alpha_\lambda(g)$,

$$\tilde{\pi}_\lambda(g) \circ \phi_\lambda = \alpha_\lambda(g) \phi_\lambda \circ \pi_\lambda(g),$$

is uniquely defined.

The gauge transformation $t(\lambda)$ also defines isomorphisms of simple K -equivariant bimodules :

$$t_{\lambda\mu} : F_{\lambda\mu} \rightarrow \tilde{F}_{\lambda\mu} : f(g) \mapsto f(gt(\lambda)),$$

where $F_{\lambda\mu}, \tilde{F}_{\lambda\mu} \subset F_{\mu\lambda}[G]$, in particular, $t_{\lambda\lambda} = t_\lambda$. As it was observed above, isomorphisms $\rho_{\lambda\mu} : F_{\lambda\mu} \rightarrow V_\lambda \otimes V_\mu^*$ and $\tilde{\rho}_{\lambda\mu} : \tilde{F}_{\lambda\mu} \rightarrow \tilde{V}_\lambda \otimes \tilde{V}_\mu^*$ of equivariant K -bimodules are also canonical. By Schur's Lemma, for fixed λ, μ we have

$$(\phi_\lambda \otimes (\phi_\mu^*)^{-1}) = C \tilde{\rho}_{\lambda\mu} \circ t_{\lambda\mu} \circ \rho_{\lambda\mu}^{-1},$$

for some constant C , whence replacing ϕ_λ by $C\phi_\lambda$ we get $\alpha_\lambda(g) = \alpha_\mu(g)$, i.e., the system $\{\phi_\lambda\}$ gives an isomorphism between the dynamical data in question. \square

Remark 6.7. In the case $A = \{1\}$ Theorem 6.6 recovers the result of [Mo], [EG], where usual twists in $k[G]$ (modulo gauge equivalence) were shown to be in bijection with single irreducible projective representations of subgroups of G (modulo conjugation).

Corollary 6.8. *A dynamical datum is minimal, i.e., $G = K$, if and only if the corresponding dynamical twist $J(\lambda)$ is minimal.*

Example 6.9. Here is an example of a dynamical datum that gives rise to a non-minimizable dynamical twist.

Let $f : A_2 \rightarrow A_1$ be an isomorphism between two abelian subgroups of G which are *not* conjugate to one another and such that for any irreducible character λ of A_1 we have $\text{Ind}_{A_1}^G \lambda \cong \text{Ind}_{A_2}^G f^* \lambda$. Then taking $A = A_1$, $K = A_2$, $V_\lambda = k_\lambda$, and $\pi_\lambda = f^* \lambda$, $\lambda \in A^*$, we get a dynamical datum $(K, \{V_\lambda\})$ that gives rise to a non-minimizable dynamical twist (since A is not contained in any subgroup conjugate to K).

Below is an example of such a situation that we learned from R. Guralnick. Take $G = S_6$, the symmetric group of degree 6, and let A_1 and A_2 be two non-cyclic subgroups of order 4 with A_1 moving precisely 4 points (in a single orbit) and A_2 having three orbits of size 2, e.g., $A_1 = \{1, (12)(34), (13)(24), (14)(23)\}$ and

$A_2 = \{1, (12)(34), (34)(56), (12)(56)\}$. Clearly, A_1 and A_2 are not conjugate as they have orbits of different size.

On the other hand, we have $\text{Ind}_{A_1}^G 1 \cong \text{Ind}_{A_2}^G 1$ since A_1 and A_2 intersect each conjugacy class in G in the same number of elements (namely, the identity and 3 conjugates of $(12)(34)$). Let λ_i denote 3 non-trivial irreducible characters of A_1 . Note that $\text{Ind}_{A_1}^G \lambda_i \cong \text{Ind}_{A_1}^G \lambda_j$ since the three proper subgroups of A_1 (and A_2) are conjugate in G . Next,

$$\text{Ind}_{\{1\}}^G 1 = \text{Ind}_{A_1}^G (\text{Ind}_{\{1\}}^{A_1} 1) = \text{Ind}_{A_1}^G 1 + \sum \text{Ind}_{A_1}^G \lambda_i \cong \text{Ind}_{A_1}^G 1 + 3 \text{Ind}_{A_1}^G \lambda,$$

where λ is any of λ_i . We have precisely the same equation for A_2 , and since $\text{Ind}_{A_1}^G 1 \cong \text{Ind}_{A_2}^G 1$ as noted above, this implies that $\text{Ind}_{A_1}^G \lambda \cong \text{Ind}_{A_2}^G f^* \lambda$.

Example 6.10. Here we describe a general construction of dynamical data with $K = A$ and compute corresponding dynamical twists in an important concrete situation. Let A be an abelian subgroup of G and $f : A^* \rightarrow A^*$ be a bijection (which is not necessarily a group isomorphism). Suppose that for all $\lambda, \mu \in A^*$ the character $f(\lambda) - f(\mu)$ is conjugate to $\lambda - \mu$ via some element $g(\lambda, \mu)$ in the normalizer $N(A)$ of A , i.e.,

$$(f(\lambda) - f(\mu))(a) = (\lambda - \mu)(\text{Ad}_{g(\lambda, \mu)} a), \quad a \in A,$$

where Ad denotes the left adjoint action of G on itself.

Take $K = A$, $V_\lambda = k$, and $\pi_\lambda = f(\lambda)$, $\lambda \in A^*$. We claim that $D_f := (K, \{V_\lambda\})$ is a dynamical datum for (G, A) . Indeed, we have $\text{Ind}_A^G(\lambda - \mu) \cong \text{Ind}_A^G(f(\lambda) - f(\mu))$ since the characters $\lambda - \mu$ and $f(\lambda) - f(\mu)$ are conjugate by $g(\lambda, \mu) \in G$.

Moreover, for any two bijections $f_1, f_2 : A^* \rightarrow K^*$ with the above property, the corresponding data D_{f_1} and D_{f_2} are isomorphic if and only if $f_1(\lambda) = \text{Ad}_g f_2(\lambda) + \text{const}$, where $g \in N(A)$. In particular, D_f is isomorphic to the trivial datum (which corresponds to the constant twist $1 \otimes 1$) if and only if it defines a minimizable twist if and only if $f(\lambda) = \text{Ad}_g \lambda + \text{const}$, $g \in N(A)$.

Choose elements $g(\lambda, \mu) \in G$ and let X be a G -module and $x \in X[\mu]$. Define a corresponding K -module homomorphism by

$$(30) \quad \Psi(\lambda, x) : V_\lambda \rightarrow V_{\lambda - \mu} \otimes X : 1 \mapsto 1 \otimes g(\lambda, \lambda - \mu)^{-1} x$$

(note that $g(\lambda, \lambda - \mu)^{-1}$ maps $X[\mu]$ to $X[f(\lambda) - f(\lambda - \mu)]$ for all $\lambda, \mu \in A^*$).

Applying the exchange construction we obtain a dynamical twist

$$(31) \quad J(\lambda) = \sum_{\mu\nu \in A^*} g(\lambda, \lambda - \mu - \nu) g(\lambda - \mu, \lambda - \mu - \nu)^{-1} P_\nu \otimes g(\lambda, \lambda - \mu - \nu) g(\lambda, \lambda - \mu)^{-1} P_\mu,$$

where $P_\mu = \frac{1}{|A|} \sum_{a \in A^*} \mu(a) a$ is the projection on $k[G][\mu]$.

As a concrete example of such a situation consider the group $G = GL_n(F_p) \ltimes F_p^n$, where F_p is the field of p elements and $A = F_p^n$ is regarded as an additive group on which $GL_n(F_p)$ acts by multiplication, and an arbitrary bijection $f : A^* \rightarrow A^*$ (note that any two non-trivial characters of A are conjugate since $GL_n(F_p)$ acts on $(F_p^n)^* \setminus \{0\}$ transitively).

We compute dynamical twist (31) when $n = 1$. In this case $G = F_p^\times \ltimes F_p$, where F_p^\times acts on $A = F_p$ by multiplication (this G can be thought as the group of affine

transformations of the line over F_p). Let ζ be a primitive p th root of unity in k and $\tau : F_p^* \rightarrow F_p$ be the isomorphism given by

$$\lambda(a) = \zeta^{\tau(\lambda)a}, \quad \lambda \in A^*, a \in A.$$

In what follows we suppress τ and identify F_p^* and F_p . For any bijection $f : F_p \rightarrow F_p$ define elements of F_p^\times :

$$(32) \quad g(\lambda, \mu) = \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, & \text{if } \lambda \neq \mu; \\ 1, & \text{otherwise,} \end{cases}$$

conjugating $\lambda - \mu$ and $f(\lambda) - f(\mu)$ for all λ, μ . Taking these $g(\lambda, \mu)$ and identifying $F_p^* \setminus \{0\}$ and $F_p \setminus \{0\}$ we get an explicit formula for a dynamical twist:

$$(33) \quad J(\lambda) = \frac{1}{p^2} \sum_{\mu\nu \in F_p^*, ab \in F_p} \zeta^{\nu a + \mu b} \frac{\nu}{f(\lambda - \mu) - f(\lambda - \mu - \nu)} \frac{f(\lambda) - f(\lambda - \mu - \nu)}{\mu + \nu} a \otimes \\ \otimes \frac{\mu}{f(\lambda) - f(\lambda - \mu)} \frac{f(\lambda) - f(\lambda - \mu - \nu)}{\mu + \nu} b,$$

where the fractions with zero denominators are replaced by 1.

7. A METHOD OF CONSTRUCTING DYNAMICAL TWISTS FOR FINITE NILPOTENT GROUPS

Here we describe how to produce dynamical data from nilpotent Lie algebras over finite fields. Let \mathfrak{g}^0 be a split semisimple Lie algebra over the field $F = F_p$ of p elements (e.g., $\mathfrak{sl}_n(F_p)$). Consider a nilpotent Lie algebra $\mathfrak{g} = \mathfrak{g}^0 t F[t]/t^{n+1}$ defined by $[xt^i, yt^j] = [x, y]t^{i+j}$ for all $x, y \in \mathfrak{g}^0$.

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus F$ be a non-trivial 1-dimensional central extension of \mathfrak{g} defined by

$$[(x, \alpha), (y, \beta)] = ([x, y], \omega(x, y)), \quad x, y \in \mathfrak{g},$$

for a 2-cocycle $\omega : \wedge^2 \mathfrak{g} \rightarrow F$ defined as follows. Let $r \in \mathfrak{h}^0$ be a regular element, set $\omega(xt^i, yt^j) = \delta_{i+j, n+1}(x, [r, y])$, where $(,)$ is the invariant scalar product in \mathfrak{g} (it is straightforward to check that ω is a 2-cocycle).

Suppose that p is big enough so that we do not have to divide by p in the Campbell-Hausdorff formula

$$(34) \quad \exp(x) \exp(y) = \exp\left(x + y + \frac{1}{2}[x, y] + \dots\right), \quad x, y \in \widehat{\mathfrak{g}},$$

where there are finitely many summands on the right hand side. Then $\widehat{G} = \exp \widehat{\mathfrak{g}} = \{\exp(x) \mid x \in \widehat{\mathfrak{g}}\}$ is a finite nilpotent group. Note that $\{\exp(\alpha) \mid \alpha \in F\}$ is a central subgroup of \widehat{G} and that the latter is a central extension of $G = \exp(\mathfrak{g})$.

The Cartan decomposition of $\mathfrak{g}^0 = \mathfrak{h}^0 \oplus \mathfrak{n}_+^0 \oplus \mathfrak{n}_-^0$ yields the following decomposition of $\widehat{\mathfrak{g}}$:

$$(35) \quad \widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-,$$

where $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus F$, $\mathfrak{h} = \mathfrak{h}^0 t F[t]/t^{n+1} \oplus F$, and $\mathfrak{n}_\pm = \mathfrak{n}_\pm^0 t F[t]/t^{n+1}$. For every $\lambda \in \mathfrak{h}^*$ let us define a functional $\phi_\lambda \in \widehat{\mathfrak{g}}^*$ by setting

$$\phi_\lambda|_{\mathfrak{n}_+ \oplus \mathfrak{n}_-} = 0, \quad \phi_\lambda(h) = \lambda(h), \quad \phi_\lambda(\alpha) = \alpha,$$

where $h \in \mathfrak{h}$, $\alpha \in F$. Let us denote $\widehat{\mathfrak{b}}_\pm = \mathfrak{n}_\pm \oplus \widehat{\mathfrak{h}}$.

Lemma 7.1. $\widehat{\mathfrak{b}}_{\pm}$ are Lie subalgebras of $\widehat{\mathfrak{g}}$ which are maximal isotropic subspaces for the alternating bilinear form

$$(36) \quad b_{\lambda}(x, y) = \phi_{\lambda}([x, y]), \quad x, y \in \widehat{\mathfrak{g}}, \lambda \in \mathfrak{h}^*.$$

Proof. It is clear from the above definitions that $\widehat{\mathfrak{b}}_{\pm}$ are isotropic subspaces for b_{λ} . Let us show that they are maximal. The dimension d of a maximal isotropic subspace of b_{λ} is given by

$$(37) \quad d = \dim \widehat{\mathfrak{g}} - \frac{1}{2} \text{rank } b_{\lambda} = \dim \widehat{\mathfrak{g}} - \frac{1}{2} \dim \mathcal{O}_{\lambda},$$

where \mathcal{O}_{λ} is the orbit of the coadjoint action of $\widehat{\mathfrak{g}}$ containing λ . We have $\dim \mathcal{O}_{\lambda} = \dim \widehat{\mathfrak{g}} - \dim \text{St}(\phi_{\lambda})$, where

$$\text{St}(\phi_{\lambda}) = \{x \in \widehat{\mathfrak{g}} \mid \text{Ad}_x \phi_{\lambda} = 0\}$$

is the stabilizer of ϕ_{λ} . Clearly, $\widehat{\mathfrak{h}} \subseteq \text{St}(\phi_{\lambda})$. In order to conclude that $d = \dim \widehat{\mathfrak{b}}_{\pm}$ it suffices to show that $\text{St}(\phi_{\lambda}) = \widehat{\mathfrak{h}}$. Note that $\text{St}(\phi_{\lambda})$ is compatible with the Cartan decomposition of $\widehat{\mathfrak{g}}$, so it is enough to prove that $\mathfrak{n}_{\pm} \cap \text{St}(\phi_{\lambda}) = 0$. Let $x \in \mathfrak{n}_{+} \cap \text{St}(\phi_{\lambda})$, where $x = x_m t^m + x_{m+1} t^{m+1} + \dots$ with $x_m \neq 0$. Let $y = b t^{n+1-m}$, where $b \in \mathfrak{n}_{-}$ is such that $(x_m, [r, b]) \neq 0$. Then $\phi_{\lambda}([x, y]) = \omega(x_m, b) \neq 0$, a contradiction that shows $\mathfrak{n}_{+} \cap \text{St}(\phi_{\lambda}) = 0$. One shows that $\mathfrak{n}_{-} \cap \text{St}(\phi_{\lambda}) = 0$ in a similar way. \square

Let $\widehat{B}_{\pm}, N_{\pm}, H, \widehat{H}$ be the subgroups of \widehat{G} corresponding to Lie subalgebras $\widehat{\mathfrak{b}}_{\pm}, \mathfrak{n}_{\pm}, \mathfrak{h}, \widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$, respectively. Clearly, ϕ_{λ} gives rise to the character X_{λ} of \widehat{B}_{+} (or \widehat{B}_{-}):

$$(38) \quad X_{\lambda}(\exp(x)) = \psi \circ \phi_{\lambda}(x), \quad x \in \widehat{B}_{\pm},$$

where $\psi : F \rightarrow k^{\times}$ is any fixed non-trivial homomorphism. The result of Kazhdan ([K], Proposition 2) together with Lemma 7.1 imply that

$$(39) \quad V_{\lambda} := \text{Ind}_{\widehat{B}_{+}}^{\widehat{G}} X_{\lambda} \cong \text{Ind}_{\widehat{B}_{-}}^{\widehat{G}} X_{\lambda}$$

is an irreducible representation of \widehat{G} for every $\lambda \in \mathfrak{h}^*$. Since $\exp F$ acts on V_{λ} as a scalar defined by ψ , we have a family of projective irreducible representations of G parameterized by \mathfrak{h}^* . Note that since ω is trivial on \mathfrak{h} we have that $\widehat{H} \cong H \oplus \mathbb{Z}/p\mathbb{Z}$.

Proposition 7.2. $\{V_{\lambda} \mid \lambda \in \mathfrak{h}^*\}$ is a minimal dynamical datum for G .

Proof. Let $\lambda, \mu \in \mathfrak{h}^*$, Y be a G -module, and k_{λ} denote a 1-dimensional \widehat{H} -module determined by X_{λ} , then we have

$$\begin{aligned} \text{Hom}_{\widehat{G}}(V_{\lambda} \otimes V_{\mu}^*, Y) &\cong \text{Hom}_{\widehat{G}}(\text{Ind}_{\widehat{B}_{+}}^{\widehat{G}} X_{\lambda} \otimes \text{Ind}_{\widehat{B}_{-}}^{\widehat{G}} X_{-\mu}, Y) \\ &\cong \text{Hom}_{\widehat{B}_{+}}(k_{\lambda} \otimes \text{Ind}_{\widehat{B}_{-}}^{\widehat{G}} X_{-\mu}, Y) \\ &\cong \text{Hom}_{\widehat{H}}(k_{\lambda} \otimes k_{-\mu}, Y) \cong Y[\lambda - \mu], \end{aligned}$$

where the second isomorphism is a consequence of the Frobenius reciprocity and the third uses that $\text{Ind}_{\widehat{B}_{-}}^{\widehat{G}} X_{\lambda}|_{\widehat{B}_{+}} \cong \text{Ind}_{\widehat{H}}^{\widehat{B}_{+}} X_{\lambda}$ (which holds since $\widehat{G} = \widehat{B}_{+}N_{-}$, $\widehat{B}_{-} = \widehat{H}N_{-}$, and $X_{\lambda}|_{N_{-}} = 1$). Therefore, $V_{\lambda} \otimes V_{\mu}^* \cong \text{Ind}_{\widehat{H}}^{\widehat{G}} X_{\lambda-\mu} \cong \text{Ind}_{\widehat{H}}^{\widehat{G}} X_{\lambda-\mu}$, as required. \square

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